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KEY FINDINGS

- Mathematically, the mass of the distribution of after-tax wealth derived from a high-volatility concentrated stock is likely to shift toward zero with the investment horizon. This puts the prospects of long-run wealth growth and preservation in serious peril. Diversifying concentrated risks is essential for avoiding catastrophic loss of wealth.
- For risk-averse investors, median wealth and mode wealth are more relevant wealth distribution statistics than mean wealth. The tax liability that would result from diversifying a concentrated low-basis stock has only a secondary effect on the median and mode of wealth. The stock's volatility, on the other hand, has a primary effect on these statistics. Therefore, for many concentrated low-basis stock investors, it is optimal to liquidate the stock and invest the after-tax proceeds in a diversified portfolio, despite the significant upfront liquidation tax burden.
- Tax-efficient techniques for disposing of concentrated low-basis stock might strike the balance between the urgency to diversify and aversion to taxes. We model a theoretical tax-free transition from a concentrated stock to a diversified portfolio. The theoretical tax-free transition can be used as the ideal limiting case in future studies of tax-efficient transition techniques.

ABSTRACT

Successful entrepreneurs and executives often hold much of their wealth in a highly appreciated single stock and thereby face a difficult financial dilemma. On the one hand, the high idiosyncratic volatility of a concentrated single stock position can lead to significant risk of catastrophic losses; on the other hand, selling the stock can result in an immediate and punitive tax burden. This article develops an analytical framework for evaluating this choice and explains how it relates to classic betting strategies and economic theory. For many investors, a full and immediate liquidation of their appreciated single stock might be optimal from the perspective of long-run wealth growth and preservation. In fact, in the absence of diversification, most investors must expect catastrophic losses of wealth over reasonable investment horizons. For investors reluctant to incur an upfront tax burden, tax-efficient techniques for disposing of an appreciated single stock might strike the balance between the urgency to diversify concentrated risk and aversion to taxes. Whereas for median and mode cumulative wealth, the primary effect likely comes from diversification, be it tax efficient or not, for mean cumulative wealth, the tax efficiency of diversification can yield a tangible improvement.

Successful entrepreneurs and executives often end up with much of their wealth concentrated in a highly appreciated single stock. As Quisenberry and Welch (2005) put it: "Despite living through dramatic market declines, horror stories from friends, colleagues, and the media, and a surfeit of market research, many investors continue to hold too much of their net worth in a single concentrated stock position." Although continuing to hold a concentrated stock might seem like a status-quo stance, it is, in fact, one of the most risky investment strategies an investor could pursue, and it is especially problematic for those wealth creators who seek to become guardians of wealth for themselves and their families.

The mention of an appreciated single stock might evoke such analogies as Google, Apple, or Amazon. However, the performance of most stocks has been, and will continue to be, a far cry from these well known but exceptionally rare successes. Past studies have used examples of individual stocks,[1] actual stock return distributions,[2] and Monte Carlo simulated return distributions[3] to demonstrate just how risky holding concentrated stock positions might be. For example, Bessembinder (2018) shows that, during the 1926–2016 period, for all the 25,967 common stocks in the Center for Research in Securities Prices (CRSP) database, by far the most frequent one-decade buy-and-hold return is –100%.[4] Here, we will see that fundamental statistics are rather ruthless toward single stock investors concerned with growth and preservation of their wealth.

This article pursues three main objectives. First, it develops simple formulas for compounded post-liquidation wealth statistics: mean, median, mode, and shortfall probability. Investors and their advisors can use these formulas to assess the risks and rewards of various investment alternatives without a need for complex computer simulations. Second, it explains how wealth distribution statistics are related to optimal betting strategies[5] and expected utility theory[6] and why focusing on arithmetic mean returns might be suboptimal for most investors, especially for investors with long investment horizons. Third, it shows what type of improvement an investor can expect from transitioning from a concentrated stock to a diversified portfolio tax-efficiently[7] rather than by liquidating the concentrated stock in a fully taxable sale and then reinvesting the after-tax proceeds in the diversified portfolio.

In dealing with appreciated single stock, many investors face one of the most consequential financial decisions of their lives. And the choice is anything but easy. On the one hand, a significant risk of catastrophic loss—a direct consequence of the high volatility of the concentrated stock position—puts the prospects of long-run wealth growth and preservation in a serious peril. On the other hand, selling the stock, fully or partially, in order to reduce the idiosyncratic volatility of the investment portfolio, results in an immediate and punitive tax burden. The decision about how to handle concentrated stock is further complicated by other factors—rational (such as insider knowledge and regulatory trading restrictions on affiliated

[1] See, for example, Stein et al. (2000), Miller (2002), and Boyle et al. (2004).

[2] See, for example, Boyle et al. (2004) and Bessembinder (2018).

[3] See, for example, Stein et al. (2000), Boyle et al. (2004), Quisenberry and Welch (2005), and Bessembinder (2018).

[4] To be precise, Bessembinder (2018) displays the frequency distribution of returns rounded to the nearest 5%.

[5] See, for example, Kelly (1956) and Ethier (2004).

[6] See, for example, Campbell and Viceira (2002).

[7] See, for example, Quisenberry and Welch (2005).

shareholders) and irrational (such as behavioral biases[8] and a sentimental value attached to the stock[9]).

It is a well-documented phenomenon in behavioral economics that people often avoid a sure loss in favor of an uncertain, but probable, bigger loss.[10] As Daniel Kahneman puts it, “People become risk seeking when all their options are bad.”[11] So it comes as no surprise that, even absent trading restrictions, many investors choose uncertain economic loss over a certain and immediate tax liability and delay diversifying their appreciated concentrated stock holdings. The gap in after-tax wealth achieved with a diversified portfolio and with a concentrated stock can be thought of as a dollar cost of failure to diversify.

Using compounded post-liquidation wealth statistics, this article shows that the benefit from reducing volatility can easily outweigh the tax burden of liquidation (and, perhaps, all other considerations as well, including expectations that the stock might have a higher mean return than a diversified index portfolio). The urgency to diversify increases with the stock’s return volatility and the investor’s risk aversion and investment horizon. Moreover, the cost of delaying the diversification grows rapidly over time. Importantly, a stock’s high arithmetic mean return is not a sufficient condition for long-run wealth preservation.

To mitigate the tax burden of diversifying an appreciated single stock, various strategies have been developed as an alternative to an outright sale.[12] These strategies are particularly beneficial in light of behavioral preferences to avoid sure losses in favor of larger, but uncertain, losses. By reducing the upfront tax cost, tax-efficient diversification strategies tilt the scale in favor of diversification and thereby improve the prospects of long-run wealth growth and preservation. A tax-free diversification modeled herein serves as an upper bound on tax-efficient diversification. Interestingly, mitigating upfront liquidation taxes has only a secondary effect on median and mode wealth: For these wealth statistics, the main improvement comes from volatility reduction offered by diversification, whether it is fully taxable, tax efficient, or completely tax free. On the other hand, tax-efficient diversification has a first-order effect on the mean level of compounded wealth, which is not affected by volatility but is reduced by tax costs.[13]

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[8] Boyle et al. (2004) offer a list of behavioral biases that, in their view, would push the investor in the direction of holding, rather than selling, a single stock: anchoring, overconfidence, attraction to long shots, underestimating the likelihood of extreme events, regret avoidance, reference dependency, loss avoidance. The latter bias refers to “incurring large risks to avoid a sure loss” in a form of a tax bill upon selling the stock. Reference dependency is epitomized by an attitude where investors “view the stock’s most recent high as its fair value” (Brunel 2006, 184, quoting Meir Statman).

[9] Brunel (2006, 184) gives two separate examples of a widow and children having difficulty parting with a concentrated position inherited from, in one case, a husband and, in the other, a father. Lucas (2020, 13) warns about potential “vacuum in purpose and family culture” and loss of “fiscal responsibility” when a family business is sold.

[10] Kahneman (2011, 278–288).

[11] Kahneman (2011, 280).

[12] Before the Taxpayer Relief Act (TRA) of 1997 “shorting-against-the-box” was a popular transaction. Although, as explained in Welch (1999), it was not a perfect hedge, it was as close to it as it gets. The TRA of 1997 introduced Internal Revenue Code Section 1259, which classified this type of transaction as a “constructive sale.” That is, for tax purposes, entering into a short-against-the-box transaction became a sale of appreciated stock. As an alternative, a spectrum of strategies that do not constitute a constructive sale (and involve various levels of economic risk) have been described in the literature. These range from exchange funds, to completion portfolios, to various hedging strategies, to charitable techniques. For detailed description and analysis of such strategies, see Welch (1999, 2002, 2003), Kiefer (2000), Miller (2002), Boyle et al. (2004), Quisenberry and Welch (2005), Brunel (2006, 176–184), Gordon (2009), and Lucas (2020, 22–24).

[13] To be clear, the purpose of this article is neither to dissuade business builders from taking entrepreneurial risks nor to explain entrepreneurial behavior. (In fact, the following calculations suggest that assuming concentrated risk might be necessary for achieving great financial wealth.) Rather, the objective is to develop mathematical tools that would help investors and their advisors analyze the trade-off between volatility and taxes, particularly *after* entrepreneurial success has been achieved and wealth creators begin to shift their focus from wealth creation to wealth preservation.

EXHIBIT 1

A Two-Period Illustration: Volatility Reduction and Wealth Preservation

	Single Stock			Index Portfolio		
	Year 1	**Year 2**	**Year-2 Post-Liquidation Wealth**	**Year 1**	**Year 2**	**Year-2 Post-Liquidation Wealth**
Initial Value			\$100			\$70
Initial Cost Basis			\$0			\$70
Return Paths						
Path 1 (Prob = 25%)	–50%	–50%	\$18	–10%	–10%	\$61
Path 2 (Prob = 25%)	–50%	60%	\$56	–10%	20%	\$74
Path 3 (Prob = 25%)	60%	–50%	\$56	20%	–10%	\$74
Path 4 (Prob = 25%)	60%	60%	\$179	20%	20%	\$92
One-Period Return Distribution						
Mean	5%	5%		5%	5%	
Std. Dev.	55%	55%		15%	15%	
Year-2 Post-Liquidation Wealth Distribution						
Mean			\$77			\$75
Median			\$56			\$74
Mode			\$56			\$74
Std. Dev.			\$61			\$11
Prob(Wealth < \$70)			75%			25%
Prob(Wealth < \$35)			25%			0%

NOTE: The capital gains tax rate is assumed to be 30%.

A TWO-PERIOD ILLUSTRATION OF THE POTENTIAL PERILS OF VOLATILITY

To set the stage, let's start with a simple two-period example that illustrates the risk of a concentrated single stock compared to a diversified portfolio. Furthermore, the example shows how the goals of reducing volatility and mitigating taxes directly clash with each other. The example is summarized in Exhibit 1.

Single Stock

The first two columns in Exhibit 1 show four alternative return paths for a hypothetical single stock. Columns 1 and 2 show returns in Year 1 and 2, respectively. Assume that over a two-year period, the returns can follow four equally likely paths: down by 50% in both years, down by 50% then up by 60%, up by 60% then down by 50%, and up by 60% in both years. These assumptions imply that, in each year, the mean stock return and the standard deviation of stock return are 5% and 55%, respectively.[14]

Column 3 shows the cumulative post-liquidation wealth of each return path at the end of the two-year period, assuming that the initial value of the stock position is \$100, its cost basis is \$0, and the capital gains tax rate is 30%.[15] Note that under

[14] The mean return is computed as $\frac{1}{4}(-50\%)+\frac{1}{4}(-50\%)+\frac{1}{4}60\%+\frac{1}{4}60\% = 5\%$ and the standard deviation of return is computed as $\sqrt{\frac{1}{4}(-50\%-5\%)^2+\frac{1}{4}(-50\%-5\%)^2+\frac{1}{4}(60\%-5\%)^2+\frac{1}{4}(60\%-5\%)^2} = 55\%$.

[15] As of October 2022, the top bracket federal tax rate on long-term capital gains was 23.8%. The assumed capital gains tax rate of 30% adds a theoretical state and local tax of 6.2%.

these assumptions, if the investor was to liquidate the stock at the beginning of the two-year period, she would be left with $70 of after-tax proceeds.

The bottom section of Exhibit 1 shows statistics for the post-liquidation wealth. Under our single stock assumptions, the mean post-liquidation wealth is $77. However, the median post-liquidation wealth (that is, the level of wealth achieved with 50% probability) is just $56. The mode of post-liquidation wealth (that is, the most likely level of wealth) is also just $56.

There is a large dispersion of outcomes across different return paths leading to a standard deviation of post-liquidation wealth of $61. Moreover, the investor ends up with less than $70 (which is the after-tax proceeds she would have obtained from liquidating the stock in the beginning of the two-year period) with a probability of 75% and with less than $35 (which is half the after-tax proceeds from liquidating the stock in the beginning of the two-year period) with a probability of 25%. As a result, although holding on to the single stock might yield a substantial upside ($179 post-liquidation on $100 invested in Path 4), the likelihood and magnitude of potential losses ostensibly make the single stock a poor candidate for wealth *preservation*.

Index Portfolio

The next three columns in Exhibit 1 show the results of a transition to a hypothetical index portfolio. In our example, the transition policy is quite simple: Liquidate the single stock at the beginning of the two-year period and invest the after-tax proceeds in the index portfolio. The value and cost basis of the index portfolio are thus both $70.

Over the two-year period, this diversified index portfolio can follow four equally likely paths of returns: down by 10% in both years, down by 10% then up by 20%, up by 20% then down by 10%, and up by 20% in both years. In each year, the mean and standard deviation of the index return are thus 5% and 15%, respectively. Note that the index portfolio and single stock have the same arithmetic mean return, but the volatility of the index portfolio return is substantially lower than that of the single stock return.

The last column in Exhibit 1 shows two-year post-liquidation wealth outcomes.[16] The mean post-liquidation wealth is similar to that obtained under the single stock investment—$75 vs. $77. The best outcome under the single stock scenario is almost twice as high as under the index portfolio scenario—$179 compared to $92. However, the lower upside of the index portfolio is compensated by a significantly lower downside risk. The median post-liquidation wealth and mode of post-liquidation wealth are now both $74, which is about one-third higher than $56 under the single stock scenario. The dispersion of post-liquidation wealth outcomes is almost six times lower than under the single stock investment—$11 compared to $61. Finally, the probability of the post-liquidation wealth falling below $75 is just 25% and the probability of it falling below $35 is now 0%.

To summarize, the distribution of wealth achieved after two periods with a highly volatile single stock investment is different from that achieved with a lower-volatility index portfolio in two ways. First, with the single stock investment, the mass of wealth distribution is shifted significantly toward zero. Second, with the single stock investment, the skewness of wealth distribution is much greater, as is evidenced by its long right tail. In other words, the example in this section shows that even though sticking with a single stock position might yield a greater upside, diversifying into an index portfolio certainly reduces the risk of a significant downside and improves

[16] Note that in the index portfolio scenario the investor pays liquidation taxes twice: first on the transition from the single stock to the index portfolio at the beginning of the two-year period and then on the liquidation of the index portfolio at the end of year two.

the chances of wealth preservation. As we show in the remainder of the article, this result is very general and becomes stronger as the investment horizon increases.

Before proceeding, it is worth pointing out that although the single stock's 55% return volatility used in the example might look high, it is not inconsistent with the experience that many entrepreneurial wealth creators might have. In an analysis omitted here for the sake of brevity, using data that go back to the 1980s, my colleagues and I found that the median annualized volatility of IPO stocks in the first five years after the IPO was in excess of 50%. The median volatility increased above 55% for smaller IPO stocks and above 60% for IPOs in the information technology (IT) and healthcare (HC) sectors. The 75th percentile of IPO stock volatility in these two sectors was as high as approximately 90% (the 75th percentile for all IPOs was lower but is still a staggering number—approximately 70%).[17]

These high volatility levels can be confirmed using risk estimates produced by the Barra risk model, again, using data that go back to the 1980s. A median Barra model volatility for IT and HC stocks in the Russell 2000 Index universe was around 55% and the 75th percentile of volatility was around 65%. For the 2,000 smallest stocks out of the top 5,000 by market capitalization, the median Barra model volatility in the IT and HC sectors was around 60% and the 75th percentile of volatility was around 70%. Based on these observations, highly volatile concentrated stock positions should be a common occurrence (and, therefore, a common problem).[18]

DISTRIBUTION-FREE RESULTS FOR COMPOUNDED WEALTH

Let us begin by stating the key property of wealth compounding (see Section A of the appendix):

When returns are independently and identically distributed (IID), mean wealth compounds with log mean return $ln(1 + E[R])$*, whereas median wealth compounds with mean log return* $E[ln(1 + R)]$.[19]

Although at first sight this might look like an arcane mathematical fact, this result is crucial for understanding the properties of long-run wealth compounding and for designing optimal strategies for growing and preserving wealth.

Three asymptotic properties of compounded wealth follow from this result (see Section A of the appendix). First, mean wealth is strictly greater than median wealth. Second, median wealth becomes infinitesimally small relative to mean wealth as the investment horizon increases.[20] Third, whereas the probability of wealth exceeding median wealth is, by definition, always 50%, the probability of wealth exceeding mean wealth converges to 0 as the investment horizon increases.

[17] Bessembinder (2018) finds similarly high levels of single stock volatility: During the 1926–2016 period, for all the 25,967 common stocks in the CRSP database, the pooled standard deviations of monthly returns and buy-and-hold annual returns were 18.1% and 81.9%, respectively. Note that the latter is almost 20% higher than the annualized standard deviation of monthly returns, which was 62.4% (18.1% times the square root of 12).

[18] Later in this article, we show how the results change when the assumed single stock return volatility increases or decreases relative to the 55% level. Qualitatively, all the conclusions remain the same even when the volatility is lower than 55%. Moreover, underestimation of the stock's volatility (for example, assuming that the volatility is 30% when the realized volatility turns out to be above 50%) can lead to devastating wealth outcomes. Therefore, concentrated stock investors and their advisors should be aware of the levels of after-tax wealth statistics for high but very realistic volatility levels.

[19] Section A of the appendix shows that the statement that mean wealth compounds with log mean return is exact for any IID returns, whereas the statement that median wealth compounds with mean log return holds asymptotically for any IID returns and exactly for IID lognormal returns.

[20] Note that whether median wealth becomes infinitesimally small in absolute terms depends on the parameters of the return distribution, as we will see later.

These three related results tell us that, in the long run, the distribution of compounded wealth has such a long right tail that mean wealth has little to do with what an investor can reasonably "expect" as the level of her long-run compounded wealth.[21] Whereas mean wealth is dominated by a few large positive outliers, the mass of the wealth distribution, which is better described by such statistics as median and mode, falls well below the mean. With this in mind, we can now turn to the relationship between return volatility and skewness of compounded wealth distribution.

LOGNORMAL COMPOUNDED WEALTH

When returns are IID with finite variance, the distribution of long-run compounded wealth converges to lognormal (see Section B of the appendix).[22] This has several important consequences:

1. *The gap between mean and median wealth increases as a function of the product of variance of investment returns and investment horizon*.
2. *Maximizing median wealth results in a growth-optimal strategy, that is, a strategy that maximizes the long-run compounded growth rate of the investor's wealth*.
3. *A growth-optimal strategy, which maximizes median wealth, is optimal for a risk-averse investor with a log-utility function (that is, with a power utility function and the coefficient of relative risk aversion of 1), whereas maximizing mean wealth is optimal for a risk-neutral investor*.
4. *An investor with a relative risk aversion coefficient three times as high as that of a log-utility investor maximizes the mode of compounded wealth (that is, the most likely level of wealth to occur)*.

Section C of the appendix shows that mean, median, and mode wealth at time T can be derived, respectively, as

$$E[W_T] = W_0 e^{\mu T}$$

$$Median[W_T] = W_0 e^{\mu T - \frac{1}{2}\sigma^2 T}$$

$$Mode[W_T] = W_0 e^{\mu T - \frac{3}{2}\sigma^2 T}$$

where μ and σ are the mean and standard deviation of the investment return and W_0 is the initial invested capital.[23] These expressions make it clear that return variance and time horizon, and more precisely, the product of the two, play a key role in the gap between mean wealth, median wealth (the level of wealth achieved with

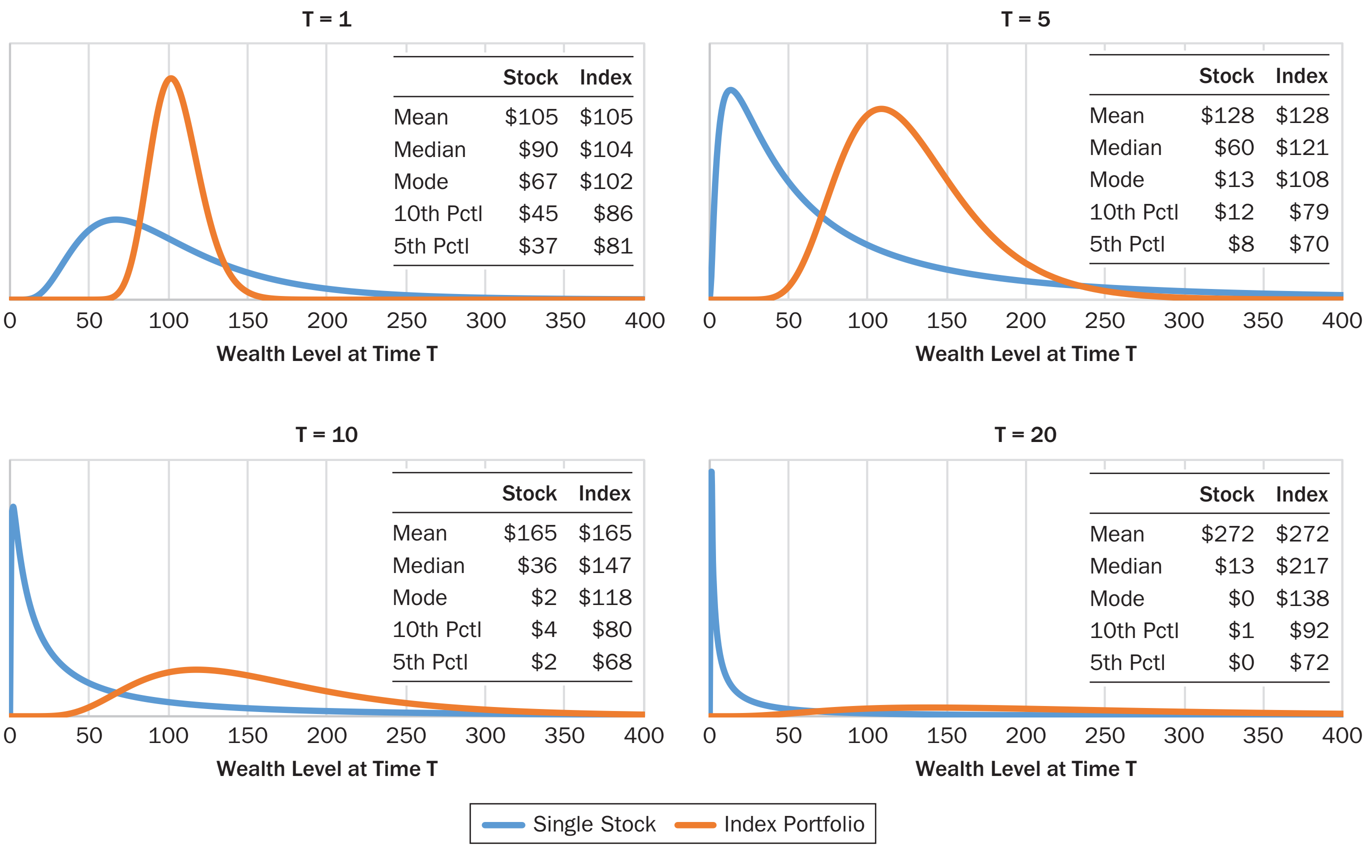
[21] This point was made in Hughson, Stutzer, and Yung (2006).

[22] Conclusions of the analysis presented here still hold qualitatively even when stock return distributions exhibit fat tails (see, for example, Kon 1984 and Koundouri, Kourogenis, and Pittis 2016). If anything, fat tails make the risk of catastrophic losses for a single stock investment even greater.

[23] To be mathematically precise (see Section C of the appendix), m should be measured as ln(1 + $E[R]$), and the asset volatilities should be measured using log returns. However, here we ignore the differences between log average return ln (1 + $E[R]$) and average return $E[R]$ and between volatilities of percent returns and log returns. This simplifies the analysis without changing qualitative conclusions. For small $E[R]$, ln (1 + $E[R]$) ≈ $E[R]$. Also, for returns measured over short time intervals, for example, daily, $\sigma \approx Var[R]$ (see Hull 2003, 238).

EXHIBIT 2

Distribution of Compounded Wealth at 1, 5, 10, and 20-Year Horizons



50% probability), and mode wealth (the most likely level of wealth to occur).[24] And it is this gap that, as we demonstrate later, really matters for the trade-off between diversification and the tax burden of liquidating a concentrated stock position; the decision to liquidate and diversify substantially changes the volatility of investment returns, which in turn has large consequences for wealth compounding, especially over long horizons.

Exhibit 2 shows the distribution of compounded wealth achieved with a hypothetical single stock and an index portfolio and reports mean, median, and mode wealth at investment horizons of 1, 5, 10, and 20 years, alternatively.[25] Both the stock and index are assumed to have 5% annual arithmetic mean return. The annual return volatility for the single stock and index portfolio is 55% and 15%, respectively. The initial invested capital is $100 for both the stock and index investment.[26]

[24] Bessembinder (2018) similarly observes that volatility drives a wedge between mean and median return.

[25] Strictly speaking, if asset returns are lognormal, the return of an index portfolio, which is a weighted average of asset returns, is not lognormal. And vice versa: if the index return is lognormal, then asset returns are not. Although there is a technical inconsistency in the assumption that the stock and index portfolio's returns are both lognormal, lognormal approximation is still adequate for the purposes of illustrating the main ideas of the article. In reality, the risk of concentrated stock positions might be far greater than a lognormal distribution could describe; a lognormal distribution does not capture the fact that there is a nonzero probability of a single stock's value declining all the way to zero.

[26] Liquidation taxes are ignored here, because this section is concerned with the effects of volatility and investment horizon on wealth accumulation. Taxes will be introduced in the next section.

As shown in Exhibit 2, at a 1-year horizon, the distribution of wealth achieved with the lower volatility index portfolio is approximately symmetric with similar mean, median, and mode wealth levels. The highly volatile single stock exhibits some degree of right skewness. As the investment horizon increases, the mass of the wealth distribution shifts to the left. This shift is particularly pronounced for the highly volatile single stock. At a 20-year horizon, mean wealth of $272 achieved by investing $100 in the single stock lies all the way in the right tail of the distribution, the mass of the wealth distribution clusters well below $50, and median and mode wealth are $13 and approximately $0, respectively. For the index portfolio, however, both median and mode wealth, which are $217 and $138, respectively, are well above the initial investment of $100. At different investment horizons, the 10th and 5th percentiles of the wealth distribution with the index portfolio remain in the $80–$90 and $70–$80 range, respectively. At the same time, with the single stock, the 10th and 5th percentiles are only $45 and $37, respectively, at a 1-year horizon and decline to close to $0 at longer horizons.[27]

These results show yet again that mean compounded wealth (and arithmetic mean return that determines its level) is not the right quantity to focus on for a long-run investor. Depending on return volatility, mean wealth might lie far out in the right tail of the wealth distribution and might be a highly unlikely outcome to achieve. The highly volatile single stock, despite having the same arithmetic mean return and achieving the same mean wealth as the index portfolio, performs disastrously poorly based on the median and mode wealth criteria.

The intuition for these results is simple. It is difficult to compound out of large negative return shocks that are typical of highly volatile assets. We have seen this in the two-period illustration in Exhibit 1: When the wealth in up-down and down-up paths 2 and 3 shrank by 50% in one of the years, even a positive return of 60% in the other year was not nearly enough to compensate the investor for this loss of wealth. You need a 100% return just to get back to zero, which implies an arithmetic mean return of 25%!

THE IMPACT OF TAXES AND INVESTOR OPTIMISM ABOUT THE CONCENTRATED STOCK

Exhibit 2 paints a grim picture for single stock investors concerned with long-run wealth growth and preservation: The mass of wealth distribution shifts relentlessly toward zero as the investment horizon increases. A lower volatility index portfolio exhibits substantially better wealth distribution properties. Taxes are often mentioned as one of the key reasons for reluctance to diversify a single stock position into an index portfolio. Another reason for delaying diversification could be investor optimism about the prospects of a concentrated stock. In this section, we begin to investigate the effects of taxes and investor optimism on the expected distribution of post-liquidation compounded wealth. Investor optimism is expressed via the single stock's alpha.

Section D of the appendix incorporates taxes and the single stock's alpha into wealth distribution statistics: A fraction of a single stock position is sold down at the beginning of the investment period, a capital gains tax is paid on the liquidation gain, and the post-liquidation proceeds of the stock sale are reinvested in an index portfolio. At the end of the investment horizon, all the positions—the remaining single

[27] Bessembinder (2018) shows similar results with real stock return data: For individual stocks, both the gap between mean and median returns and the positive skewness of return distribution are large and increasing with investment horizon.

stock and the index—are fully liquidated, liquidation capital gains tax is paid, and the post-liquidation wealth statistics are calculated and reported.[28]

Note that here the basis step-up at death is ignored. Concentrated wealth is often a problem faced by ultra-high-net-worth families for whom estate tax exemption might shield (from estate tax) only a small part of their wealth. Such families typically transfer their wealth into trusts that shield the wealth from estate tax but do not allow for a basis step-up at death.[29] Economically, it makes sense to give up the basis step-up in favor of avoiding estate tax. First, estate tax rates are substantially higher than long-term capital gain rates. Second, whereas realization of capital gains can be deferred, the same cannot be said about death (at least as of the time of this writing).

Let's assume that the market arithmetic mean return is 5%, the interest rate is 0%, the stock's beta is 1.0, the market return volatility is 15%, and the stock's residual return volatility is 52.9%. The stock's alpha will be assumed to be either 0% or 10%, alternatively. Under these assumptions (see Section D of the appendix for details), the mean return and volatility of the index portfolio are 5% and 15%, respectively, and the volatility of the single stock is 55%. These values are exactly the same as those used in the examples in the previous two sections. The single stock's arithmetic mean return is either 5% for the 0% alpha or 15% for the 10% alpha.[30]

Let's further assume that the initial single stock position has a value of $100, cost basis of $0, and capital gains tax rate of 30% in every period. For example, if the investor liquidates $10 worth of the stock, she realizes a $10 capital gain (because the cost basis is $0), pays $3 in capital gains taxes, and invests the after-tax proceeds of $7 in the index portfolio. Section D of the appendix shows how to compute the mean return and volatility of a portfolio that allocates part of its capital to a single stock and part to an index portfolio.

Single Stock with 0% Alpha

Let's start by assuming that the single stock's alpha is 0%. The stock position is partially liquidated at the beginning of the investment period at $10 intervals. The relationship between the upfront tax burden of liquidation and the end-of-period post-liquidation wealth distribution statistics is shown in Exhibit 3, Panel A, and loss probabilities are shown in Exhibit 3, Panel B.

For example, when none of the stock position is liquidated, and thus the upfront tax burden is $0, Exhibit 3, Panel A, shows that the 10-year mean, median, and mode post-liquidation wealth is $115, $25, and $1, respectively (see marker labels in the exhibit). When half of the position is liquidated, the upfront tax burden is $15 and the 10-year mean, median, and mode post-liquidation wealth is $109, $65, and $27, respectively.

Exhibit 3, Panel B, shows that the probability of the 10-year post-liquidation wealth falling below $25 is as high as 50% when the investor holds on to the single stock

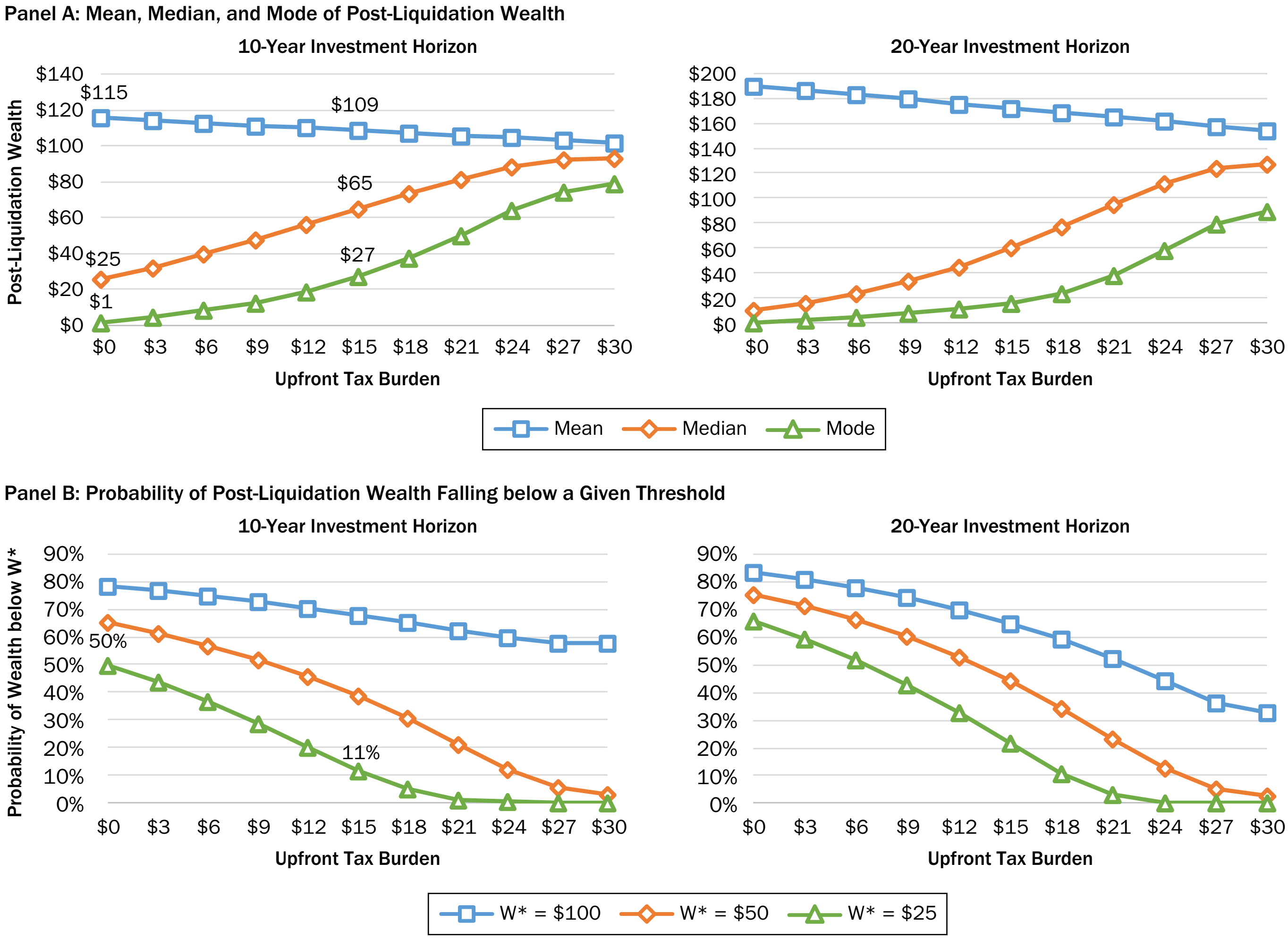
[28] For the sake of simplicity, dividends are ignored, because the level and volatility of dividend income (and of taxes associated with it) are typically small relative to stock price average appreciation and volatility.

[29] See, for example, Sosner, Liberman, and Liu (2021) and references therein for further details on estate tax planning.

[30] Throughout the article, we explore long-run wealth accumulation. Pastor and Stambaugh (2012) pointed out that, according to conventional wisdom, the annualized volatility of stock returns is lower in the long run than in the short run due to mean reversion. Indeed, they find that mean reversion reduced long-run volatility. However, they show that uncertainty about various parameters more than offsets the volatility-reducing effect of mean reversion, such that long-run volatility is, in fact, significantly higher than short-run volatility. The main contributor to long-run volatility is uncertainty about future expected returns, especially when expected returns are persistent.

EXHIBIT 3

Distribution Statistics for Post-Liquidation Wealth When the Stock's Alpha Is 0%



NOTE: The upfront tax burden reported on the horizontal axis is computed as the portion of the stock position sold at time 0, times the assumed capital gains tax rate of 30%.

position but is reduced to just 11% when half of the position is diversified into the index portfolio (and the upfront tax burden of $15 is incurred). It is reduced all the way to 0% when the single stock position is fully liquidated and the after-tax proceeds are invested in the index portfolio.

Exhibit 3 shows that by diversifying, the investor reduces mean post-liquidation wealth. In this case, the reduction occurs only because of the upfront tax burden of liquidation.[31] This is because the arithmetic mean returns of the stock and the index are the same. At the same time, by diversifying the single stock position, the investor significantly increases median and mode wealth. The probability of a substantial loss of wealth—that is, the end-of-period post-liquidation wealth falling below $50 or $25—is very high for large allocations to the single stock but is reduced to close to 0% for large allocations to the index portfolio.

To summarize, when arithmetic mean returns of the single stock and index portfolio are equal, diversifying away from a highly volatile single stock significantly improves

[31] See Section E in the appendix for proof of this result.

the outcome for the investor based on the criteria we discussed in the previous section—median wealth and mode wealth. In addition, diversifying away from a single stock reduces the probability of significant loss of wealth. Importantly, this conclusion holds in the presence of taxes incurred on the upfront sale of the single stock position.

Single Stock with 10% Alpha

Let's now assume that the single stock's alpha is 10% per year and that this alpha is maintained for the entirety of the investment horizon. Under the assumptions described previously—mean market return of 5%, risk free rate of 0%, and the stock's beta of 1.0—this results in the single stock's arithmetic mean return of 15%, or three times as high as that of the index. Moreover, the increase in the stock's mean return comes from alpha and is not associated with any additional market risk exposure, which makes this high return even more attractive for the investor.

Exhibit 4, Panel A, shows that mean post-liquidation wealth obtained with the single stock declines as the allocation of after-tax proceeds to the index portfolio increases. Two factors contribute to this result. First, the single stock has an arithmetic mean return three times as high as that of the index portfolio. Second, the upfront tax burden of liquidating the stock reduces the size of the initial investment.[32] It is important to keep in mind, however, that, as shown in Exhibit 5, the probability of achieving the mean level of wealth with the single stock is 19% after 10 years and is just 11% after 20 years, whereas the probability of achieving mean wealth with the index portfolio remains at approximately 40%.

Exhibit 4, Panel A, further shows median and mode wealth, which, as we discussed previously, are more relevant statistics for understanding the prospects of long-run wealth growth and preservation than the mean wealth. The median post-liquidation wealth is maximized at both the 10-year and the 20-year horizons when approximately 70% of the $100 single stock position is liquidated upfront at a tax cost of $21. The mode of post-liquidation wealth is maximized at both the 10-year and 20-year horizons when approximately 90% of the $100 single stock position is liquidated upfront at a tax cost of $27. It is instructive to see that *even when the arithmetic mean return of the stock is three times as high as that of the index*, the investor maximizing median wealth or mode wealth should still liquidate most of the single stock position despite the significant upfront tax burden.

Exhibit 4, Panel B, shows that the probability of substantial loss of wealth—for example, wealth being reduced to less than $50 or $25 at the end of the investment horizon—remains quite high with the single stock, even when its arithmetic mean return is three times as high as that of the index portfolio. Liquidating a fraction of the single stock position and reinvesting the after-tax proceeds in an index portfolio goes a long way toward reducing the probability of a catastrophic loss of wealth. For example, when more than 70% of the stock position is liquidated (at the tax cost of $21 or more), the probability of wealth falling below $25 at both the 10-year and 20-year investment horizons is reduced to 0%.

Finally, we could ask the question: *How high does the single stock's alpha need to be to make the investor indifferent about whether to continue to hold the stock or diversify into the index portfolio?* Under the parameter values used in this section, for an investor concerned with median wealth at 10-year or 20-year horizons, the investor must expect the stock's alpha to be approximately 13%. However, in this case, the optimal level of diversification—that is, the level of diversification that yields maximum median wealth—is around 60%: The investor should liquidate $60 worth of the stock position out of $100, pay a liquidation tax of $18, and invest the after-tax proceeds of

[32] See Section E of the appendix for proof of this result.

EXHIBIT 4

Distribution Statistics for Post-Liquidation Wealth When the Stock's Alpha Is 10%

NOTE: The upfront tax burden reported on the horizontal axis is computed as the portion of the stock position sold at time 0, times the assumed capital gains tax rate of 30%.

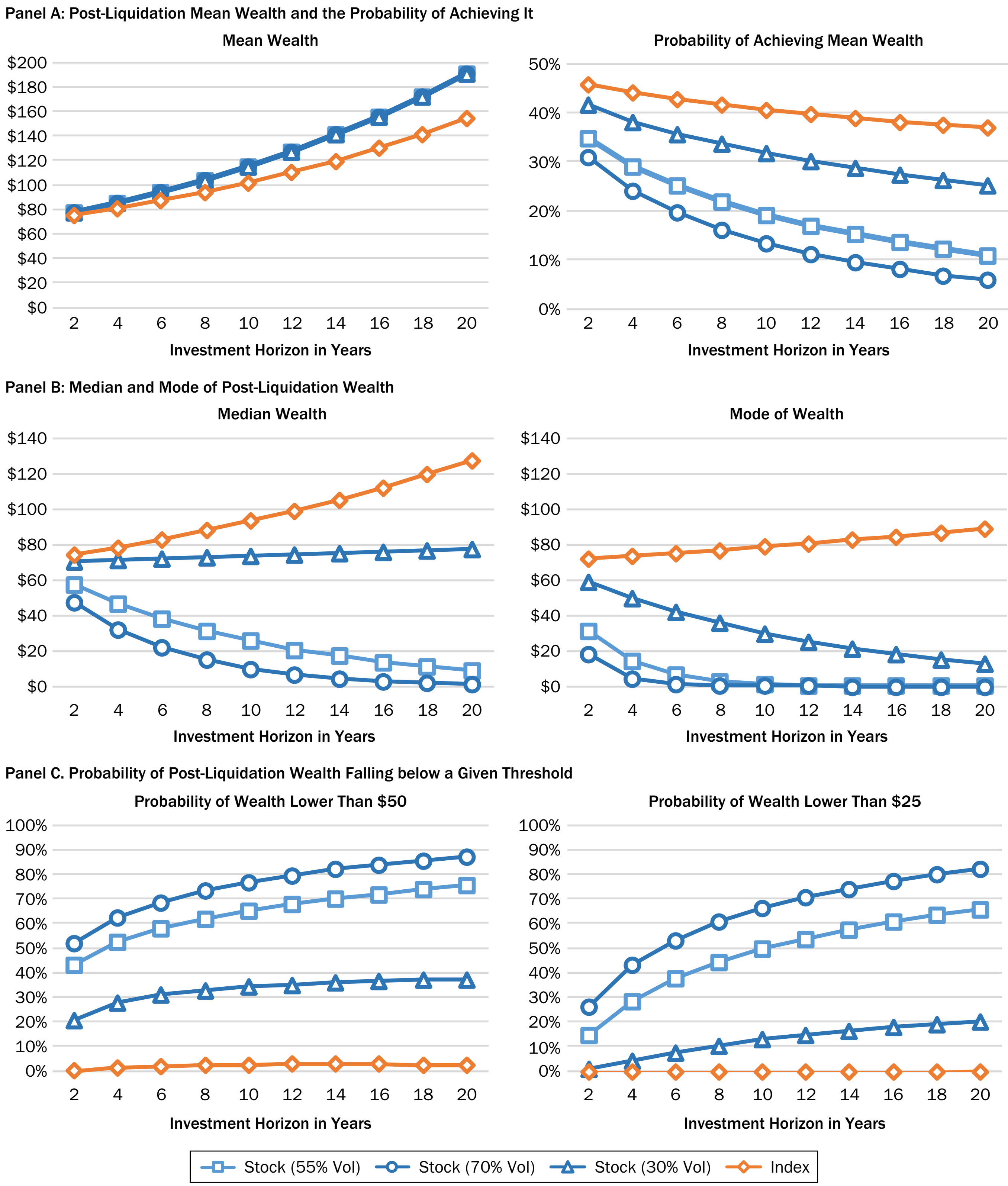
$42 in the index portfolio. For the investor concerned with mode wealth, the stock's alpha must be approximately 42%. The level of diversification that maximizes mode wealth is again around 60%.

EFFECTS OF INVESTMENT HORIZON AND VOLATILITY

The results in the previous section demonstrate that an investor concerned with long-run wealth growth and preservation should diversify a substantial portion of her single stock position into an index portfolio, even when the investor believes that the single stock offers meaningfully higher expected returns than the market. In this section, we compare two edge-case investment policies—continuing to hold a single stock and fully diversifying the single stock into an index portfolio—and measure the effects of the investment horizon and the single stock's volatility on post-liquidation wealth. The investment horizon varies from 2 to 20 years, and the single stock's volatility varies from 55% to 70% and 30%, alternatively. The single stock's alpha

EXHIBIT 5

Distribution Statistics for Post-Liquidation Wealth at Different Investment Horizons and Levels of Single Stock's Volatility



and beta are assumed to be 0% and 1.0, respectively, which implies that the stock's arithmetic mean return is the same as that of the index. The rest of the parameters remain the same as in the previous section: The mean index return is 5%, the risk-free rate is 0%, and the index return volatility is 15%. All the formulas used in the calculations are derived in Section E of the appendix.

The results are shown in Exhibit 5. Note that although investment in the stock starts with $100, investment in the index portfolio starts with $70: The stock position is fully liquidated at time 0, and out of $100 of proceeds, $30 are set to aside to pay a capital gains tax on liquidation and $70 are reinvested in the index portfolio.

Exhibit 5 shows that under plausible assumptions about return distribution and parameter values, holding on to the single stock creates a significant drag on long-run wealth compounding, in terms of a lower probability of achieving mean wealth, lower levels of median wealth and mode wealth, and a higher probability of catastrophic loss of wealth. These deleterious effects of holding on to the single stock increase with the volatility of the stock and the investment horizon.

These results are particularly important in light of conventional wisdom that investors benefit from the compounding of wealth over long horizons. This insight is indeed true for the diversified portfolio. In Exhibit 5, with a diversified index portfolio, mean, median, and mode wealth increase with investment horizon, and the probability of a catastrophic loss is virtually nil. However, the idea of long-run compounding completely misses its mark for investors with a concentrated single stock. For a single stock, as the investment horizon increases, the mass of the wealth distribution shifts rather rapidly to the left, with median and mode wealth falling close to zero and the probability of massive losses of wealth climbing to well above 50%. Put differently, wealth compounding is about dollars, not mean returns. Volatility drives a wedge between dollar wealth and mean returns. Once a large negative dollar shock occurs and wealth declines to close to zero, it is almost impossible to climb out of that hole through mere compounding. High volatility and a longer investment horizon make precipitous drops in wealth more likely to occur, thus pushing the mass of wealth distribution toward zero.

THE COST OF PROCRASTINATION

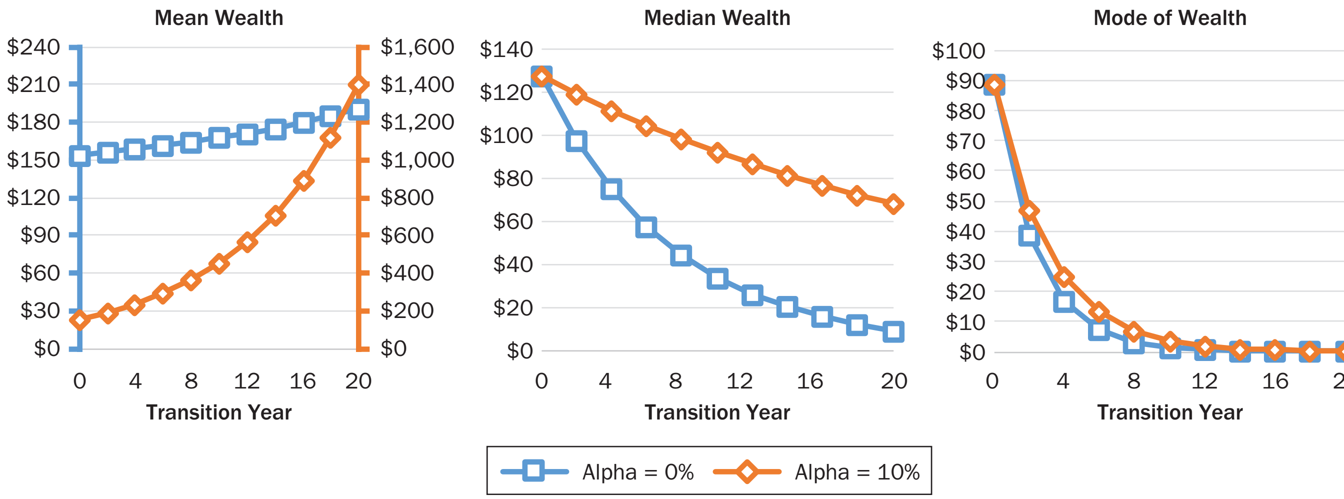
How does postponing diversification of a single stock into an index portfolio affect the distribution of investor's long-run post-liquidation wealth? To answer this question, Section F of the appendix derives formulas for mean, median, and mode wealth as a function of waiting time before the investor transitions from a single stock to an index portfolio. The transition is assumed to be full and taxable. That is, at a given time, the investor fully liquidates the appreciated stock position, pays the resulting capital gains tax, and invests the after-tax proceeds in the index portfolio. The index portfolio is held until the end of the investment horizon, at which point it is fully liquidated and the after-tax wealth is calculated.

As before, assume that the single stock position has a value of $100 and a cost basis of $0. The capital gains tax rate is 30%. The return volatility is 55% and 15% for the single stock and the index portfolio, respectively. The index portfolio arithmetic mean return is 5%. The single stock's beta is 1.0 and alpha is, alternatively, 0% or 10%, which implies an arithmetic mean return of 5% and 15%, respectively.

Exhibit 6 considers an investment horizon of 20 years. The horizontal axes in the charts show the year when the transition occurs: Year 0 corresponds to an immediate upfront transition, whereas Year 20 corresponds to holding on to the single stock and never transitioning to the index portfolio. The vertical axes show the levels of the three post-liquidation wealth statistics at the end of a 20-year investment horizon.

EXHIBIT 6

Post-Liquidation Wealth at a 20-Year Investment Horizon for Different Years of Transition from the Single Stock to the Index Portfolio



The mean wealth chart of Exhibit 6 shows that post-liquidation mean wealth increases as the transition to the index portfolio is delayed. When arithmetic mean returns of the single stock and index portfolio are the same (see the left-hand axis), the effect comes from delaying the tax burden of transition. When the arithmetic mean return of the stock is significantly higher than that of the index portfolio (see the right-hand axis), delaying the transition further increases post-liquidation mean wealth due to holding on to an asset with a higher arithmetic mean return.

The results look very different for post-liquidation median and mode wealth. Both decline with the time of transition from the single stock to the index portfolio. For median wealth, a higher alpha of the single stock partially slows this decline. However, for mode wealth, a higher alpha has very little effect on the pattern of decline. As previously discussed, because median and mode wealth are more relevant statistics for long-run wealth growth and preservation, we can view the precipitous decline in these wealth statistics as the cost of delaying the transition—the cost of procrastination.

A THEORETICAL TAX-FREE LIQUIDATION

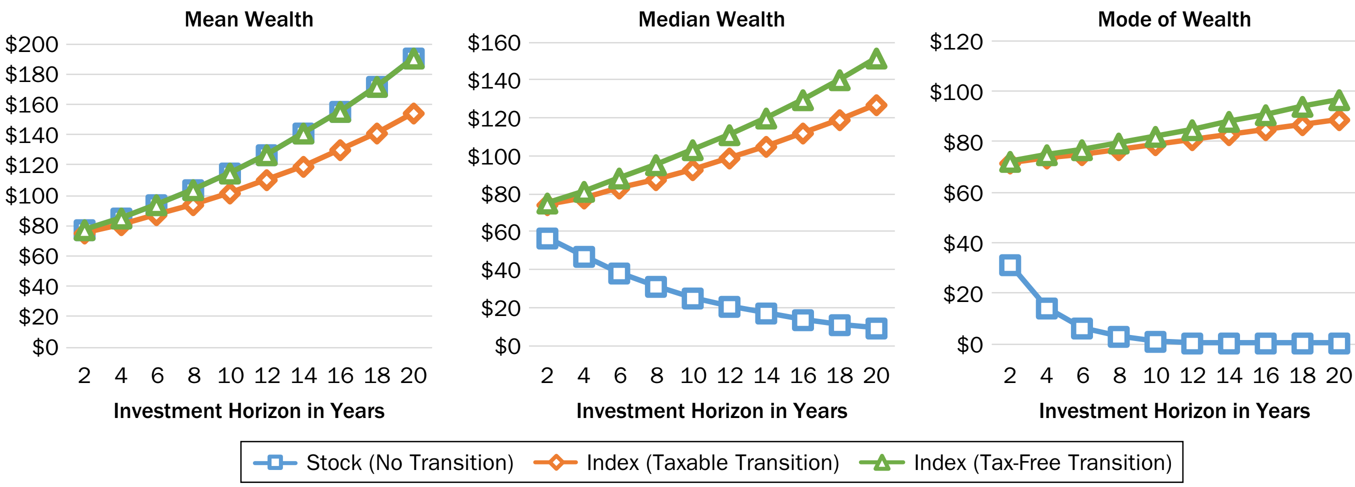
Now imagine that the investor can execute a tax-free transition. That is, the single stock position could be "replaced" with the index portfolio without a liquidation tax. Although with some advanced tools an investor can make a step in this direction,[33] a complete and immediate tax-free stock-for-index swap is highly unlikely. Nonetheless, as we show shortly, the theoretical example of tax-free transition is instructive for understanding the effects of diversification of concentrated risk on post-liquidation wealth. In essence, by taking taxes off the table, a tax-free transition allows us to understand why investors might be so reluctant to diversify, even when, from the point of view of wealth growth and preservation, immediate upfront diversification might be a far superior choice than hanging on to a concentrated stock.

Mathematically, a tax-free transition means swapping out the single stock mean return and volatility for the index portfolio mean return and volatility without incurring

[33] See, for example, Quisenberry and Welch (2005).

EXHIBIT 7

A Theoretical Tax-Free Transition to the Index Portfolio



any liquidation tax burden (see Section G of the appendix). Let's use the same parameter values as before: The single stock position has a value of \$100 and a cost basis of \$0; the capital gains tax rate is 30%; the return volatility is 55% and 15% for the single stock and index portfolio, respectively; and the arithmetic mean return of the stock and index is identical at 5%.

Exhibit 7 plots mean, median, and mode wealth at different investment horizons for a single stock with 55% return volatility. It shows the wealth statistics for a fully taxable transition and for a tax-free transition from the stock to the index portfolio. Both the fully taxable and the tax-free transition occur at the start of the investment period at time 0. The results for continuing to hold the stock (that is, no transition) and for the fully taxable transition have already been shown in Exhibit 5. They are repeated in Exhibit 7 for the ease of comparison with the tax-free transition.

The leftmost chart of Exhibit 7 shows the mean post-liquidation wealth. Because arithmetic mean returns of the single stock and index portfolio are the same and there is no upfront tax burden for the tax-free transition, the tax-free transition makes the post-liquidation wealth with the index portfolio identical to the post-liquidation wealth with the single stock. The initial tax savings from the tax-free transition create a significant increase in the long-run mean wealth compared to the fully taxable transition. Moreover, as we have seen in Exhibit 5, Panel A, the probability of achieving this wealth, which is only a function of return volatility, is twice as high under the index portfolio scenario as under the single stock scenario at the 10-year investment horizon and is almost four times as high at the 20-year investment horizon.

Mitigating the upfront tax burden through the tax-free transition also increases the median and mode of the post-liquidation wealth. However, the first-order effect on the median and mode of the post-liquidation wealth distribution comes from shifting the allocation from a highly volatile single stock to a lower-volatility index portfolio, regardless of whether it is done in a fully taxable or tax-free transition.

To summarize, a tax-free transition from a concentrated single stock to a diversified index portfolio captures the best of both worlds: On one hand, like a fully taxable transition, it offers far better prospects of wealth growth and preservation (captured by median wealth and mode wealth) than the single stock but without an upfront tax bite. On the other, compared to a fully taxable transition, it substantially improves the upside (captured by mean wealth), especially, at longer investment horizons.

CONCLUSION

Volatility is perilous for the long-run growth and preservation of wealth. The conventional wisdom that by taking the long view investors can enjoy wealth compounding over multiple decades does not apply to highly volatile concentrated stock positions—the risk of destroying the wealth is simply too high. Is an investor then better off liquidating the stock, paying the capital gains tax, and investing the after-tax proceeds in a diversified index portfolio? It depends. Investor-specific characteristics (risk aversion and investment horizon), stock-specific characteristics (volatility and mean return), current and expected future tax rates, and the investor's access to tax-efficient transition methods affect this calculus. By putting together the relevant parameters of the problem, the formulas derived in this article allow investors and their advisors to evaluate the trade-off between the benefit of diversifying the risk of an appreciated concentrated stock position and the tax consequences of doing so. The analytical framework developed here provides guidance on the appropriate level of diversification of a concentrated single stock position.

Calculations suggest that many investors might indeed discover that, at least in theory, a full and immediate liquidation of their appreciated single stock position is the optimal choice. As we have seen, reducing volatility has a first-order effect on the long-run growth and preservation of wealth. In practice, however, investors might be reluctant to incur the tax burden of liquidating the stock. Tax-efficient techniques for disposing of appreciated single stock described in prior literature[34] might strike the balance between the urgency to diversify and aversion to taxes. Compared to a fully taxable liquidation, a tax-efficient liquidation improves the upside, especially, at longer investment horizons.

APPENDIX

DISTRIBUTION OF COMPOUNDED WEALTH

A. Properties of Compounded Wealth under IID Returns

Let the Initial investment be W_0. Given a series of investment returns $\{R_t\}$ at time T, the compounded wealth generated by the investment is

$$W_T = W_0 \prod_{t=1}^{T}(1+R_t) = W_0 e^{\sum_{t=1}^{T} r_t} \tag{A1}$$

where r_t is the log return defined as $r_t \equiv \ln(1+R_t)$.

Hughson, Stutzer, and Yung (2006) show that, under the assumption that the returns are independently and identically distributed (IID), the mean of compounded wealth, defined in Equation A1, is given by

$$E[W_T] = W_0 e^{\ln(1+E[R])T} \tag{A2}$$

Furthermore, using the results in Ethier (2004), they show that, if the skewness of the return is finite, median wealth converges to

$$Median[W_T] \to W_0 e^{E[\ln(1+R)]T} = W_0 e^{E[r]T} \tag{A3}$$

as T increases.

[34] See Welch (1999, 2002, 2003), Kiefer (2000), Miller (2002), Boyle et al. (2004), Quisenberry and Welch (2005), Brunel (2006, 176–184), Gordon (2009), and Lucas (2020, 22–24).

As a result, as T increases, the ratio of median wealth in Equation A3 to mean wealth in Equation A2 converges to

$$\frac{Median[W_T]}{E[W_T]} \rightarrow e^{[E[r]-\ln(1+E[R])]T} \tag{A4}$$

From Jensen's inequality, $E[r] < \ln(1+E[R])$, therefore, the ratio of median to mean wealth converges to 0 with T.

Moreover, whereas the probability of wealth W_T exceeding the median level of wealth is, by definition, 50%, Hughson, Stutzer, and Yung (2006) show that the probability of wealth W_T exceeding mean wealth, derived in Equation A2, converges to 0 as T increases.

B. Lognormal Distribution as a Limiting Distribution of Compounded Wealth

The lognormal distribution plays a special role as a limiting (or asymptotic) distribution of cumulative wealth when returns have finite variance. To see that, let's continue with the assumption we made in the previous section that the sequence of log returns $\{r_t\}$ is IID with a finite variance $\sigma^2 \equiv \mathrm{Var}[r] < \infty$. Then, by the Lindenberg–Levy central limit theorem

$$\sqrt{T}\left(\frac{1}{T}\sum_{t=1}^{T} r_t - E[r]\right) \xrightarrow{d} N(0,\sigma^2) \tag{B1}$$

Equation B1 can be rewritten as

$$\sum_{t=1}^{T} r_t \xrightarrow{d} N(E[r]T, \sigma^2 T) \tag{B2}$$

Taking logs of both sides of Equation A1 in Section A above, we obtain

$$\ln(W_T) = \ln(W_0) + \sum_{t=1}^{T} r_t \xrightarrow{d} N(\ln(W_0) + E[r]T, \sigma^2 T) \tag{B3}$$

That is, the lognormal distribution is the limiting distribution of compounded wealth.

This leads to a number of important results. First, under the lognormal distribution, the log of mean return and mean log return exhibit the following relationship (see Campbell and Viceira 2002, 26)

$$\ln(1+E[R]) = E[r] + \frac{1}{2}\sigma^2 \tag{B4}$$

where $\sigma^2 \equiv \mathrm{Var}[r]$, as defined previously. The relationship between median and mean wealth described in Equation A4 in Section A thus becomes

$$\frac{Median[W_T]}{E[W_T]} = e^{-\frac{1}{2}\sigma^2 T} \tag{B5}$$

In other words, when the return distribution is lognormal, which is the limiting distribution of IID returns with finite variance, the gap between mean and median wealth increases in return volatility and investment horizon.

Second, Ethier (2004) shows that, due to convergence in distribution to lognormality, maximizing median wealth for any distribution with finite variance and skewness corresponds to the Kelly (1956) system of proportional betting that maximizes the long-run geometric growth rate of wealth. That is, under plausible assumptions about return distributions, asymptotic lognormality implies that maximizing median wealth results in a growth-optimal strategy.

Third, Campbell and Viceira (2002, 27) show that under the lognormal return distribution, maximizing the expected power utility of wealth, $U(W) = \frac{W^{1-\gamma} - 1}{1-\gamma}$, where γ is the coefficient of relative risk aversion, corresponds to

$$\max\left(E[r] + \frac{1}{2}(1-\gamma)\sigma^2\right) = \max\left(\ln(1+E[R]) - \frac{\gamma}{2}\sigma^2\right) \tag{B6}$$

where the equality in Equation B6 follows from Equation B4. From Equation A2 in Section A, when $\gamma = 0$, that is, the investor is risk neutral, maximizing the expression in B6 corresponds to maximizing mean wealth $W_0 e^{\ln(1+E[R])T}$. From Equations A3, when $\gamma = 1$, that is, the investor has log utility of wealth,[35] maximizing the expression in B6 corresponds to maximizing median wealth $W_0 e^{E[r]T}$. This means that maximizing expected log utility corresponds to a growth-optimal strategy under the Kelly (1956) system.

Finally, as Campbell and Viceira (2002, 27) point out, a growth-optimal strategy is only optimal for a log-utility investor with $\gamma = 1$.[36] When $\gamma < 1$, a higher-risk strategy is optimal. When $\gamma > 1$, a lower-risk strategy is optimal. For example, as we demonstrate shortly, under lognormal returns, maximizing power utility with $\gamma = 3$ corresponds to maximizing the mode of wealth.

C. Properties of Lognormal Distribution and Compounded Wealth

Let's continue with the limiting lognormal distribution. Equation A1 in Section A defines compounded wealth as $W_T = W_0 e^{\Sigma_{t=1}^{T} r_t}$. Let the log return r_t be IID normally distributed with mean $\mu - \frac{\sigma^2}{2}$ and variance σ^2 (see Hull (2003, 234)). From these assumptions

$$E[r] \equiv \mu - \frac{\sigma^2}{2} \tag{C1}$$

and, using Equation B4 in Section B,

$$\ln(1+E[R]) = \left(\mu - \frac{\sigma^2}{2}\right) + \frac{1}{2}\sigma^2 = \mu \tag{C2}$$

Substituting Equations C1 and C2 into Equations A2 and A3, we obtain the following expressions for mean wealth at time T

$$E[W_T] = W_0 e^{\mu T} \tag{C3}$$

and for median wealth at time T

$$Median[W_T] = W_0 e^{\mu T - \frac{1}{2}\sigma^2 T} \tag{C4}$$

Note that because returns are IID, $E[W_T] = W_0(1+E[R])^T$ (see Hughson, Stutzer, and Yung 2006), which is exactly what we obtain when we substitute Equation C2 into C3. Furthermore, using properties of lognormal distribution, the mode of wealth is given by

[35] Using L'Hospital's rule: $\lim_{\gamma\to 1} \frac{W^{1-\gamma} - 1}{1-\gamma} = \lim_{\gamma \to 1} \frac{\partial(W^{1-\gamma} - 1)/\partial\gamma}{\partial(1-\gamma)/\partial\gamma} = \ln(W)$.

[36] This was originally discussed in Samuelson (1971, 1979).

$$Mode[W_T] = W_0 e^{\mu T - \frac{3}{2}\sigma^2 T} \tag{C5}$$

Note that, as mentioned in Section B, maximizing mode wealth corresponds to the coefficient of relative risk aversion γ being equal to 3.

Equations C3 to C5 show that, although mean wealth compounds with the arithmetic average return, for a risky investment, both median and mode wealth compound at a rate slower than that and are negatively affected by return volatility.

As the volatility and investment horizon increase, the probability of achieving mean wealth converges to 0. This result can be easily derived from the properties of lognormal distribution as follows. The cumulative distribution function is

$$CDF[z] = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{\ln(z) - \mu T + \frac{\sigma^2}{2}T}{\sqrt{2T}\sigma}\right) \tag{C6}$$

Therefore, the probability that $z > z^*$ is given by

$$Prob[z > z^*] = 1 - CDF[z^*] = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{\ln(z^*) - \mu T + \frac{\sigma^2}{2}T}{\sqrt{2T}\sigma}\right) \tag{C7}$$

Substituting wealth and mean wealth at time T for z and z^*, respectively, into Equation C7, we obtain

$$Prob[W_T > E[W_T]] = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\frac{\mu T - \mu T + \frac{\sigma^2}{2}T}{\sqrt{2T}\sigma}\right) = \frac{1}{2} - \frac{1}{2}\,\mathrm{erf}\left(\sigma\sqrt{\frac{T}{8}}\right) \tag{C8}$$

As T and σ increase, $\mathrm{erf}\left(\sigma\sqrt{\frac{T}{8}}\right)$ converges to 1, and thus the probability of exceeding mean wealth derived in Equation C8 converges to 0.

Furthermore, from Equation C6, the probability of compounded wealth at time T falling below a threshold W^* is given by

$$Prob[W_T < W^*] = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{\ln\left(\frac{W^*}{W_0}\right) - \mu T + \frac{\sigma^2}{2}T}{\sqrt{2T}\sigma}\right) \tag{C9}$$

D. Impact of Taxes on Compounded Wealth Distribution

Let's now incorporate taxes into the analysis. Assume that a fraction θ of the original single stock position can be replaced at time 0 by an index portfolio and that all the investments are liquidated at time T. Let B_0 and W_0 be the cost basis and the fair market value of the single stock position, respectively, and $t_{G,0}$ be the capital gains tax rate at time 0. The value of the remaining single stock position is given by

$$W_{0,S}(\theta) = W_0(1 - \theta) \tag{D1}$$

The after-tax capital allocated to the index portfolio is

$$W_{0,I}^X(\theta) = W_0\theta - (W_0\theta - B_0\theta)t_{G,0} = W_0\theta(1 - t_{G,0}) + B_0\theta t_{G,0} \tag{D2}$$

The total after-tax invested capital is composed of the remaining single stock and the new index portfolio

$$W_0^X(\theta) = W_{0,S}(\theta) + W_{0,I}^X(\theta) \tag{D3}$$

Substituting Equations D1 and D2 into Equation D3 and simplifying, we obtain

$$W_0^X(\theta) = W_0 - (W_0 - B_0)\theta t_{G,0} \tag{D4}$$

That is, capital invested at time 0 is the value of the single stock position adjusted for the liquidation tax resulting from liquidating a fraction θ of the single stock position.

The pre-liquidation value of the compounded wealth at time T is

$$W_T(\theta) = W_0^X(\theta)e^{\Sigma_{t=1}^{T} r_t(\theta)} \tag{D5}$$

where $r(\theta)$ is the return on a buy-and-hold portfolio, which at time 0 allocates a share $\frac{W_{0,S}(\theta)}{W_0^X(\theta)}$ to the single stock and a share $\frac{W_{0,I}^X(\theta)}{W_0^X(\theta)}$ to the index portfolio. The liquidation tax at time T is

$$X_T(\theta) = (W_T(\theta) - (W_{0,I}^X(\theta) + B_0(1-\theta)))t_{G,T} \tag{D6}$$

where $t_{G,T}$ is the capital gains tax rate at time T. Substituting Equation D4 into D6, we obtain

$$X_T(\theta) = (W_T(\theta) - B_0 - (W_0 - B_0)\theta(1 - t_{G,0}))t_{G,T} \tag{D7}$$

The post-liquidation wealth at time T is thus

$$W_T^X(\theta) = W_T(\theta) - X_T(\theta) \tag{D8}$$

Substituting Equations D7 into D8, we obtain

$$W_T^X(\theta) = W_T(\theta)(1 - t_{G,T}) + (B_0 + (W_0 - B_0)\theta(1 - t_{G,0}))t_{G,T} \tag{D9}$$

The second term in Equation D9 is effectively a tax credit that the investor enjoys at liquidation for the cost basis of the investment at time 0. Let's define it as

$$C_T(\theta) \equiv (B_0 + (W_0 - B_0)\theta(1 - t_{G,0}))t_{G,T} \tag{D10}$$

The mean, median, and mode of wealth in Equations C3, C4, and C5 in Section C can be updated for taxes as

$$E[W_T^X(\theta)] = W_0^X(\theta)e^{\mu(\theta)T}(1 - t_{G,T}) + C_T(\theta) \tag{D11}$$

$$Median[W_T^X(\theta)] = W_0^X(\theta)e^{\mu(\theta)T - \frac{1}{2}\sigma(\theta)^2 T}(1 - t_{G,T}) + C_T(\theta) \tag{D12}$$

$$Mode[W_T^X(\theta)] = W_0^X(\theta)e^{\mu(\theta)T - \frac{3}{2}\sigma(\theta)^2 T}(1 - t_{G,T}) + C_T(\theta) \tag{D13}$$

Finally, the probability of post-liquidation wealth falling below a threshold W^* is given by

$$Prob[W_T^X(\theta) < W^*] = \frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(\frac{\ln\left(\frac{W^* - C_T(\theta)}{W_0^X(\theta)(1 - t_{G,T})} \right) - \mu(\theta)T + \frac{\sigma(\theta)^2}{2}T}{\sqrt{2T}\sigma(\theta)} \right) \quad \text{(D14)}$$

Parameters of Return Distribution

Let the risk-free rate be r_f, the market mean return be r_m, the market volatility be σ_m, and the single stock's alpha, beta, and residual volatility be α, β, and σ, respectively. Assume that the index portfolio replicates the market. The vector of annual mean returns is then given by

$$\mu \equiv \begin{bmatrix} \mu_S \\ \mu_I \end{bmatrix} = \begin{bmatrix} \alpha + r_f + \beta(r_m - r_f) \\ r_m \end{bmatrix} \quad \text{(D15)}$$

the annual return covariance matrix is given by

$$\Sigma \equiv \begin{bmatrix} \sigma_S^2 & \sigma_{S,I} \\ \sigma_{S,I} & \sigma_I^2 \end{bmatrix} = \begin{bmatrix} \beta^2\sigma_m^2 + \sigma^2 & \beta\sigma_m^2 \\ \beta\sigma_m^2 & \sigma_m^2 \end{bmatrix} \quad \text{(D16)}$$

and the vector of weights of the single stock and the index portfolio is given by

$$\omega(\theta) \equiv \begin{bmatrix} w_S(\theta) \\ w_I(\theta) \end{bmatrix} = \begin{bmatrix} \frac{W_{0,S}(\theta)}{W_0^X(\theta)} \\ \frac{W_{0,I}^X(\theta)}{W_0^X(\theta)} \end{bmatrix} \quad \text{(D17)}$$

where μ_S and μ_I are the mean returns of the stock and the index portfolio; σ_S^2, σ_I^2, and $\sigma_{S,I}$ are the variance of the stock return, variance of the index return, and the covariance between the two, respectively; and $W_{0,S}(\theta)$, $W_{0,I}^X(\theta)$, and $W_0^X(\theta)$ are as defined in Equations D1, D2, and D4 above.

Using Equations D15, D16, and D17, we can calculate the annual mean and standard deviation of the portfolio return as

$$\mu(\theta) = \omega(\theta)^T \mu \quad \text{(D18)}$$

and

$$\sigma(\theta) = \sqrt{\omega(\theta)^T \Sigma \omega(\theta)} \quad \text{(D19)}$$

E. Edge-Case Scenarios: Holding on to the Single Stock vs. Fully Liquidating It at Time 0

Let's consider two edge-case scenarios: (1) the single stock is held until time T, that is, $\theta = 0$, and (2) the single stock is fully sold at time 0, that is, $\theta = 1$, and the after-tax proceeds of the sale are invested in an index portfolio. As before, all the investments are liquidated at time T. Let's assume, for simplicity, that $t_{G,0} = t_{G,T} = t_G$ and $B_0 = 0$.

In the case of holding on to the single stock, Equations D11 to D14 in Section D above become, respectively,

$$E[W_T^X(\theta = 0)] = W_0(1 - t_G)e^{\mu_S T} \tag{E1}$$

$$Median[W_T^X(\theta = 0)] = W_0(1 - t_G)e^{\mu_S T - \frac{1}{2}\sigma_S^2 T} \tag{E2}$$

$$Mode[W_T^X(\theta = 0)] = W_0(1 - t_G)e^{\mu_S T - \frac{3}{2}\sigma_S^2 T} \tag{E3}$$

$$Prob[W_T^X(\theta = 0) < W*] = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\ln\left(\frac{W*}{W_0(1 - t_G)}\right) - \mu_S T + \frac{\sigma_S^2}{2}T}{\sqrt{2T}\sigma_S}\right) \tag{E4}$$

In the case of full liquidation at time 0, Equations D11 to D14 become, respectively,

$$E[W_T^X(\theta = 1)] = W_0(1 - t_G)((1 - t_G)e^{\mu_I T} + t_G) \tag{E5}$$

$$Median[W_T^X(\theta = 1)] = W_0(1 - t_G)\left((1 - t_G)e^{\mu_I T - \frac{1}{2}\sigma_I^2 T} + t_G\right) \tag{E6}$$

$$Mode[W_T^X(\theta = 1)] = W_0(1 - t_G)\left((1 - t_G)e^{\mu_I T - \frac{3}{2}\sigma_I^2 T} + t_G\right) \tag{E7}$$

$$Prob[W_T^X(\theta = 1) < W*] = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\ln\left(\frac{W* - W_0(1 - t_G)t_G}{W_0(1 - t_G)^2}\right) - \mu_I T + \frac{\sigma_I^2}{2}T}{\sqrt{2T}\sigma_I}\right) \tag{E8}$$

Note that when mean returns of the single stock and the index portfolio are equal and are greater than 0, that is, $\mu_S = \mu_I = \mu > 0$, $E[W_T^X(\theta = 0)] > E[E_T^X(\theta = 1)]$. To see that, note that, since $\mu > 0$,

$$e^{\mu T} > 1 \tag{E9}$$

From Equation E9

$$0 > t_G - t_G e^{\mu T} \tag{E10}$$

Adding $e^{\mu T}$ to both sides of Equation E10 and simplifying, we obtain

$$e^{\mu T} > (1 - t_G)e^{\mu T} + t_G \tag{E11}$$

Multiplying both sides of Equation E11 by $W_0(1 - t_G)$, we obtain

$$W_0(1 - t_G)e^{\mu T} > W_0(1 - t_G)((1 - t_G)e^{\mu T} + t_G) \tag{E12}$$

Replacing the left-hand side and right hand-side using Equations E1 and E5, we conclude that

$$E[W_T^X(\theta = 0)] > E[W_T^X(\theta = 1)] \tag{E13}$$

In other words, for equal and positive mean returns, mean wealth obtained from holding the single stock is strictly higher than mean wealth obtained from liquidating the stock and investing the after-tax proceeds in the index portfolio.

F. Effects of Procrastination

Let $0 \le h \le T$ be the time when the investor transitions from the single stock to the index investment. Let $r_{S,t}$ and $r_{I,t}$ be the single stock and the index returns, respectively. Then, the compounded wealth at time h is given by

$$W_h = W_0 e^{\Sigma_{t=1}^{h} r_{S,t}} \tag{F1}$$

where $\Sigma_{t=s+1}^{s} r_{S,t} = 0$ for all s.

Given the single stock basis B_0, the post-liquidation wealth at time h is

$$W_h^X = W_h - (W_h - B_0) t_{G,h} \tag{F2}$$

where $t_{G,h}$ is the capital gains tax at time h.

The pre-liquidation wealth at time T as a function of the transition time h is given by

$$W_T(h) = W_h^X e^{\Sigma_{t=h+1}^{T} r_{I,t}} \tag{F3}$$

and the post-liquidation wealth at time T as a function of h is given by

$$W_T^X(h) = W_T(h) - (W_T(h) - W_h^X) t_{G,T} \tag{F4}$$

where $t_{G,T}$ is the capital gains tax at time T.

Let's make the following simplifying assumptions: $t_{G,t} = t_G$ for all t and $B_0 = 0$. Under these assumptions, Equations F2, F3, and F4 become

$$W_h^X = W_0 e^{\Sigma_{t=1}^{h} r_{S,t}} (1 - t_G) \tag{F5}$$

$$W_T(h) = W_0 e^{\Sigma_{t=1}^{h} r_{S,t} + \Sigma_{t=h+1}^{T} r_{I,t}} (1 - t_G) \tag{F6}$$

$$\begin{aligned} W_T^X(h) &= W_0 (1 - t_G)\left(e^{\Sigma_{t=1}^{h} r_{S,t} + \Sigma_{t=h+1}^{h} r_{I,t}} (1 - t_G) + e^{\Sigma_{t=1}^{h} r_{S,t}} t_G\right) \\ &= W_0 (1 - t_G) e^{\Sigma_{t=1}^{h} r_{S,t}} \left(e^{\Sigma_{t=h+1}^{T} r_{I,t}} (1 - t_G) + t_G\right) \end{aligned} \tag{F7}$$

From here

$$E[W_T^X(h)] = W_0 (1 - t_G) e^{\mu_S h} (e^{\mu_I (T-h)} (1 - t_G) + t_G) \tag{F8}$$

$$Median[W_T^X(h)] = W_0 (1 - t_G) e^{\left(\mu_S - \frac{1}{2}\sigma_S^2\right) h} \left(e^{\left(\mu_I - \frac{1}{2}\sigma_I^2\right)(T-h)} (1 - t_G) + t_G\right) \tag{F9}$$

$$Mode[W_T^X(h)] = W_0 (1 - t_G) e^{\left(\mu_S - \frac{3}{2}\sigma_S^2\right) h} \left(e^{\left(\mu_I - \frac{3}{2}\sigma_I^2\right)(T-h)} (1 - t_G) + t_G\right) \tag{F10}$$

Note that when $h = T$, that is, the single stock is not liquidated until the final period T, Equations F8 to F10 reduce to Equations E1 to E3 in Section E, whereas when $h = 0$, that is, the single stock is fully liquidated at time 0, Equations F8 to F10 reduce to Equations E5 to E7.

G. A Theoretical Tax-Free Transition from a Singe Stock to an Index Portfolio

Under a theoretical tax-free transition, there is no upfront liquidation tax at time 0. The investment begins with the amount of capital W_0 and compounds with the index portfolio returns. As a result, we can obtain compounded wealth distribution statistics by replacing single stock return mean and variance with index portfolio return mean and variance in Equations E1 to E3 in Section E above as follows:

$$E[W_T^X] = W_0(1-t_G)e^{\mu_I T} \tag{G1}$$

$$Median[W_T^X] = W_0(1-t_G)e^{\mu_I T-\frac{1}{2}\sigma_I^2 T} \tag{G2}$$

$$Mode[W_T^X] = W_0(1-t_G)e^{\mu_I T-\frac{3}{2}\sigma_I^2 T} \tag{G3}$$

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